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THE FLOW FIELD INDUCED BY AN OSCILLATING FLUID DROP IMMERSED IN ANOTHER FLUID

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Abstract—The flow field induced by a translatory oscillating spherical drop immersed in another fluid is considered. It is assumed that the amplitude of the oscillation is small compared with the radius of the drop. We are concerned, for the most part, with the case of a small frequency parameter M. Of particular interest is the steady streaming induced both inside and outside of the drop. The problem has been solved on the basis of the Navier–Stokes equations by the method of matched asymptotic expansions.

1. INTRODUCTION

The fluid motions produced by the migration of a liquid drop through a quiescent viscous fluid have been the subject of numerous papers. The investigations of this phenomenon which occurs in a wide variety of physico-chemical processes such as extraction and atomization have a considerable importance.

The translation of a fluid drop was first treated independently by Rybczynski (1911) and Hadamard (1911) under the assumption that the drop remained exactly spherical. Since interfacial tension acting at the junction between the two immisible fluids tends to maintain a spherical shape against the shearing stresses which are inclined to deform it, the investigations of the shape of a viscous drop settling through a fluid appear to be the most detailed treatment of a free surface. This was attempted by Saito (1913), but in view of an error in his work, Taylor & Acrivos (1964) reexamined this problem. Some investigators introduce a simplification of the boundary conditions by assuming that one side of the free surface is bounded by a fluid of extremely small viscosity. Various models (Levich 1959, Scriven 1960) to describe the rheology of interfaces have been proposed. Many have stressed the importance of the intense local pressures and temperatures. Others, (Lamb 1945, Chandrasekhar 1961) have given attention to effects such as shape–mode resonances and their associated violent mechanical deformations at the bubble surface. Nybord (1953) has called attention to still another effect which may explain the action of bubbles in certain instances. This is the presence of small-scale acoustic streaming generated by bubble-scattered sound waves. It has been experimentally established that when sound sources oscillate in a viscous fluid, a steady streaming motion will be induced. Such steady motion is called acoustic streaming and arises from the interaction of viscosity with the nonlinear inertia terms. Elder (1959) examined experimentally the steady streaming in the neighbourhood of a small bubble attached to a vibrating piston.

In a recent paper Davidson & Riley (1971) have considered the flow induced by an isolated spherical bubble which performs translational harmonic oscillations relative to the liquid in which it is embedded. Their attention was focused upon the microstreaming which is induced.

In the present paper an examination of the flow field induced by a translatory oscillating fluid drop immersed in another fluid is undertaken. Of particular interest is the steady streaming induced both inside and outside the drop. We assume that the fluids are immiscible and the particle is sufficiently small that the liquid drop maintains a fixed spherical shape. So we neglect the effects such as shape–mode resonances and their associated violent mechanical deformation at the globulan surface. We also do not consider the effects of gravity and transport across the interface due to static diffusion or diffusion associated with the microstreaming.

2. FORMULATION OF THE PROBLEM

Consider a spherical fluid drop with radius α. Adopt spherical polar coordinates at the center of the globule with the axis $0z$ pointing upstream parallel to the transverse vibration with speed $U_\infty \cos \omega t$ at infinity. Dimensionless variables will be employed throughout the analysis, and physical parameters pertaining to the interior of the drop will be distinguished from the corresponding exterior parameters by a caret. With U_∞ as a typical velocity, ω^{-1} as a typical time, a as a geometrical length and ν and $\hat{\nu}$ as the kinematic viscosities of the internal and external media respectively there are several length scales associated with this problem. From these length scales we can construct the following dimensionless parameters which characterize the motion:

$$Re = \frac{U_\infty a \rho}{\mu'}, \quad \hat{R}e = \frac{U_\infty a \hat{\rho}}{\hat{\mu}'}, \quad |M|^2 = \frac{\omega a^2}{\nu}, \quad |\hat{M}|^2 = \frac{\omega a^2}{\hat{\nu}},$$

$$\epsilon = \frac{U_\infty}{\omega a}.$$

We shall be concerned entirely with the situation in which the amplitude of the oscillation is small compared with a, so that

$$\epsilon \ll 1 \quad \text{and} \quad Re \ll |M|^2.$$

The governing equations of motion in terms of the stream functions χ and $\hat{\chi}$ have the following form:

$$\frac{1}{|M|^2} D^4\chi = \epsilon\left[\frac{1}{r^2}\cdot\frac{\partial(\chi, D^2\chi)}{\partial(r,\mu)} + \frac{2}{r^2}\cdot D^2\chi\left(\frac{\mu}{(1-\mu^2)}\cdot\frac{\partial\chi}{\partial r} + \frac{1}{r}\cdot\frac{\partial\chi}{\partial\mu}\right)\right] + \frac{\partial}{\partial\tau}(D^2\chi) \qquad [1]$$

for the exterior region $1 \leq r < \infty$ and

$$\frac{\kappa}{\gamma|M|^2} D^4\hat{\chi} = \epsilon\left[\frac{1}{r^2}\cdot\frac{\partial(\hat{\chi}, D^2\hat{\chi})}{\partial(r,\mu)} + \frac{2}{r^2}\cdot D^2\hat{\chi}\left(\frac{\mu}{(1-\mu^2)}\cdot\frac{\partial\hat{\chi}}{\partial r} + \frac{1}{r}\cdot\frac{\partial\hat{\chi}}{\partial\mu}\right)\right] + \frac{\partial}{\partial\tau}(D^2\hat{\chi}) \qquad [2]$$

for the region inside the globule.

Here $\mu = \cos\theta$, $D^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{(1-\mu^2)}{r^2}\cdot\frac{\partial^2}{\partial\mu^2}$, $Re = \frac{U_\infty a}{\nu}$

is the Reynolds number and M, κ and γ are respectively the frequency parameter, the ratio of the viscosity of the interior to the exterior fluid and the ratio of the density of the interior to that of the exterior fluid.

The radial and transverse velocity components u and v are related to χ by

$$u = \frac{1}{r^2\sin\theta}\cdot\frac{\partial\chi}{\partial\theta}, \quad v = -\frac{1}{r\sin\theta}\cdot\frac{\partial\chi}{\partial r}. \qquad [3]$$

The boundary condition satisfied by χ as $\tau\to\infty$ is

$$\chi \sim \frac{1}{2} r^2(1-\mu^2)\,e^{i\tau}. \qquad [4]$$

The boundary conditions at the interface are:

(1) Zero values of the normal velocity both inside and outside of the globule

$$\hat{\chi} = 0, \qquad \chi = 0 \qquad \text{at } r = 1. \tag{5}$$

(2) Continuity of tangential velocity across the interface, whereupon

$$\frac{\partial \hat{\chi}}{\partial r} = \frac{\partial \chi}{\partial r} \qquad \text{at } r = 1. \tag{6}$$

(3) Continuity of shear stresses of the two fluids across the interface

$$\mu'\left(\frac{\partial}{\partial r}\left(\frac{1}{r^2}\cdot\frac{\partial \chi}{\partial r}\right)\right] = \hat{\mu}'\left[\frac{\partial}{\partial r}\left(\frac{1}{r^2}\cdot\frac{\partial \hat{\chi}}{\partial r_{\text{at}}}\right)\right]_{r=1}, \tag{7}$$

These conditions hold when fluids are immiscible, the surface tension is constant, the surface viscosity effects are negligible and axial symmetry is postulated. It should be remarked at this point that in [4] there is another condition requiring the balance of normal stresses when the drop changes its form and size. One can use this condition to determine the shape of the fluid drop as a function of Weber number We, κ, γ and Re. Uniqueness of the solution for small Re is ensured by the additional conditions that the velocity be a finite continuous function of position at the center $r = 0$:

$$\hat{\chi} = \frac{\partial \hat{\chi}}{\partial r} = 0 \quad \text{at} \quad r = 1. \tag{8}$$

For the regions $r \geqslant 1$ and $r \leqslant 1$ we seek the solutions respectively of the form

$$\chi = \chi_1 + \epsilon\chi_2 + \cdots, \tag{9}$$

$$\hat{\chi} = \hat{\chi}_1 + \epsilon\hat{\chi}_2 + \cdots. \tag{10}$$

3. CONSTRUCTION OF THE SOLUTION

Substituting [9] and [10] in [1] and [2] respectively and equating the terms free of ϵ on both sides we have

$$||M|^2 \cdot \frac{\partial}{\partial \tau}(D^2\chi_1) = D^4\chi_1 \tag{11}$$

and

$$|M|^2 \cdot \frac{\gamma}{\kappa} \cdot \frac{\partial}{\partial \tau}(D^2\hat{\chi}_1) = D^4\hat{\chi}_1. \tag{12}$$

The boundary conditions for χ_1 and $\hat{\chi}_1$ are [4]–[8]. We assume that

$$\chi_1 = f(r, |M|)(1 - \mu^2)\, e^{i\tau},$$

and

$$\hat{\chi}_1 = \hat{f}(r, |M|)(1 - \mu^2)\, e^{i\tau}.$$

Then from [11] and [12] we find

$$f(r, |M|) = a_1 r^2 + a_2 \cdot \frac{1}{r} + \frac{b_1}{M^2}\left(\frac{1}{r} - M\right) e^{M_r} + \frac{b_2}{M^2}\left(\frac{1}{r} + M\right) e^{-M_r}, \tag{13}$$

$$\hat{f}(r,|M|)=\bar{a}_1 r^2+\bar{a}_2\cdot\frac{1}{r}+\frac{\bar{b}_1}{M^2}\cdot\frac{\kappa}{\gamma}\left(\frac{1}{r}-\sqrt{\frac{\gamma}{\kappa}}M\right)e^{M_r}+\frac{\bar{b}_2}{M^2}\cdot\frac{\kappa}{\gamma}\left(1+\sqrt{\frac{\gamma}{\kappa}}M\right)e^{-M_r}, \qquad [14]$$

where a_i, b_i, $\bar{a}_i$, $\bar{b}_i$ $(i=1,2)$ are constants.

Applying boundary conditions [4]–[8] we obtain the following exact solutions of [11] and [12]:

$$f(r,|M|)=\frac{1}{2}\cdot r^2-\frac{1}{2r}+\frac{3}{2M^2}\left[\left(\frac{1}{r}+M\right)e^{-M(r-1)}-\frac{1}{r}(1+M)\right]\cdot\left\{1-(1+M)\left[\left(-3+3\sqrt{\frac{\gamma}{\kappa}}M\right.\right.\right.$$
$$\left.-\frac{\gamma}{\kappa}M^2\right)e^{M\sqrt{(\gamma/\kappa)}}+\left(3+3\sqrt{\frac{\gamma}{\kappa}}M+\frac{\gamma}{\kappa}M^2\right)e^{-M\sqrt{(\gamma/\kappa)}}\Big]:\Big[\Big(6\kappa-6\sqrt{\gamma\kappa}M+3\gamma M^2-\frac{\gamma^{3/2}}{\sqrt{\kappa}}M^3$$
$$-9+9\sqrt{\frac{\gamma}{\kappa}}M-\frac{3\gamma}{\kappa}M^2-3M+3\sqrt{\frac{\gamma}{\kappa}}M^2-\frac{\gamma}{\kappa}M^3\Big)e^{M\sqrt{(\gamma/\kappa)}}+\Big(-6\kappa-6\sqrt{\gamma\kappa}M-3\gamma M^2$$
$$\left.\left.-\frac{\gamma^{3/2}}{\sqrt{\kappa}}M^3+9+9\sqrt{\frac{\gamma}{\kappa}}M+\frac{3\gamma}{\kappa}M^2+3M+3\sqrt{\frac{\gamma}{\nu}}M^2+\frac{\gamma}{\kappa}M^3\Big)e^{-M\sqrt{(\gamma/\nu)}}\right]\right\}, \qquad [13']$$

$$\hat{f}(r,|M|)=\left\{-\frac{3}{2}(1+M)\left[\left(1-\sqrt{\frac{\gamma}{\kappa}}M\right)e^{\sqrt{(\gamma/\kappa)}M}-\left(1+\sqrt{\frac{\gamma}{\kappa}}M\right)e^{-\sqrt{(\gamma/\kappa)}M}\right]r^2\right.$$
$$\left.+\frac{3}{2}(1+M)\left[\left(\frac{1}{\tau}-\sqrt{\frac{\gamma}{\kappa}}\mu\right)e^{\sqrt{(\gamma/\kappa)}M_r}-\left(\frac{1}{r}+\sqrt{\frac{\gamma}{\kappa}}M\right)e^{-\sqrt{(\gamma/\kappa)}M_r}\right]\right\}: \qquad [14']$$
$$:\left\{\left[6\kappa-6\sqrt{\gamma\kappa}M+3\gamma M^2-\frac{\gamma^{3/2}}{\sqrt{\kappa}}M^3-9+9\sqrt{\frac{\gamma}{\kappa}}M-\frac{3\gamma}{\kappa}M^2-3M\right.\right.$$
$$\left.+3\sqrt{\frac{\gamma}{\kappa}}M^2-\frac{\gamma}{\kappa}M^3\right]e^{M\sqrt{(\gamma/\kappa)}}+\left[-6\kappa-6\sqrt{\gamma\kappa}M-3\gamma M^2-\frac{\gamma^{3/2}}{\sqrt{\kappa}}M^3\right.$$
$$\left.\left.+9+9\sqrt{\frac{\gamma}{\kappa}}M+3\frac{\gamma}{\kappa}M^2+3M+3\sqrt{\frac{\gamma}{\kappa}}M^2+\frac{\gamma}{\kappa}M^3\right]e^{-M\sqrt{(\gamma/\kappa)}}\right\}.$$

We expect χ_2 to contain a term independent of τ in addition to the oscillatory one:

$$\chi_2(r,\mu,|M|,\tau)=\chi_2^{(s)}(r,\mu,|M|)+\chi_2^{(u)}(r,\mu,|M|,\tau) \qquad [15]$$

where the superscript (s) denotes steady and (u) unsteady.

The reason that a steady streaming may be induced by a periodic motion of the liquid drop is the fact that the pertubation equations $O(\epsilon)$ contain forcing terms $\propto\cos^2\tau=\frac{1}{2}(1+\cos 2\tau)$. The equations for the unsteady and steady parts of χ_2 are

$$\frac{1}{|M|^2}D^4\chi_2^{(u)}-\frac{\partial}{\partial r}(D^2\chi_2^{(u)})=\frac{1}{2r^2}\left[\frac{\partial(\chi_1,D^2\chi_1)}{\partial(r,\mu)}+2D^2\chi_1\cdot L\chi_1\right]^{(u)} \qquad [16a]$$

$$\frac{1}{|M|^2}D^4\chi_2^{(s)}=\frac{1}{2r^2}\left[\frac{\partial(\chi_1,D^2\chi_1)}{\partial(r,\mu)}+2D^2\chi_1\cdot L\chi_1\right]^{(s)} \qquad [16b]$$

where

$$L=\frac{\mu}{(1-\mu^2)}\cdot\frac{\partial}{\partial r}+\frac{1}{r}\cdot\frac{\partial}{\partial\mu}.$$

The complicated form both of the solution [13′] and the equations [16a, b] has precluded the possibility of a solution for χ_2 in closed form for all values of M. That is why furthermore we shall suppose that $|M|\ll 1$. Now we can expand the perturbation functions in [9] and [10] as a series in powers of M. For χ_1 and $\hat{\chi}_1$ we have

$$\chi_1 = \frac{(1-\mu^2)}{4}\left[2r^2 - \frac{3\kappa+2}{\kappa+1}r + \frac{\kappa}{\kappa+1}\cdot\frac{1}{r}\right]e^{i\tau} + \frac{|M|(1-\mu^2)}{6}\cdot\frac{(3\kappa+2)}{(\kappa+1)^2}\left[-\frac{3\kappa+2}{2}r + \frac{\kappa}{2r} + (\kappa+1)r^2\right]e^{-(\tau+\pi/\tau)} - \frac{|M|^2(1-\mu^2)\,e^{i(\tau+\pi/2)}}{1008(\kappa+1)^3}\Big\{63(3\kappa+2)(\kappa+1)^2r^3 - 56(3\kappa+2)^2(\kappa+1)r^2 + [12\gamma(\kappa+1)+17(3\kappa+2)(9\kappa^2+5+12\kappa)]r - [12\gamma(\kappa+1)+7(3\kappa+2)(3\kappa^2+2\kappa+3)]\frac{1}{r}\Big\} \quad [17]$$

$$\hat{\chi}_1 = -\frac{(1-\mu^2)(r^2-r^4)}{4(\kappa+1)}e^{i\tau} - \frac{|M|(1-\mu^2)(3\kappa+2)(r^2-r^4)\,e^{-(\tau+\pi/\gamma)}}{12(\kappa+1)^2} + \frac{(1-\mu^2)|M|^2\,e^{i(\tau+\pi/2)}}{1008\kappa(\kappa+1)^3} \times\cdot[9\gamma(\kappa+1)^2r^6 - 2(3\kappa+2)(3\gamma\kappa+3\gamma+17\kappa)r^4 + (9\gamma+30\gamma\kappa+21\gamma^2\kappa+78\kappa^2+56\kappa)r^2]. \quad [18]$$

In this diffuse case ($|M| \ll 1$) for the stream function χ one can develop solutions in an "inner" (Stokes) region of scale $O(a)$ and a much larger "outer" (Oseen) region of scale $O(a|M|)$. For the outer expansions we assume

$$\Psi = \Psi_1 + \epsilon\Psi_2 + \cdots \quad [19]$$

The inner [9] and the outer [19] solutions must match in the overlapped region. We concentrate first on the solution in the inner region.

Since $|M| \ll 1$ we write

$$\chi_2 = \chi_{20} + |M|\chi_{21} + |M|^2\chi_{22}. \quad [20]$$

If we substitute [20] into [16] and equate coefficients of like powers of $|M|$ we have

$$D^4\phi_{20} = 0; \quad D^4\phi_{21} = 0.$$

It may be verified *a posteriori*, using matching of the inner and outer solutions, that

$$\phi_{20} = 0, \qquad \phi_{21} = 0$$

are the only possible solutions.

According to [15] we write

$$\phi_{22} = \phi_{22}^{(0)}(r,\mu) + \phi_{22}^{(2)}(r,\mu)\,e^{2i\tau}. \quad [21]$$

The functions $\phi_{22}^{(j)}$ ($j = 0, 2$) must satisfy the equation

$$D^4\phi_{22}^{(j)} = -\frac{3\mu(1-\mu^2)}{8}\cdot\frac{(3\kappa+2)}{(\kappa+1)}\cdot\frac{1}{r^3}\left[2r - \frac{3\kappa+2}{\kappa+1} + \frac{\kappa}{\kappa+1}\cdot\frac{1}{r^2}\right]. \quad [22]$$

The solution of this equation has the form

$$\phi_{22}^{(j)} = \left\{a_1{}^j + a_2{}^jr^3 + a_3{}^jr^5 + a_4{}^jr^{-2} + \frac{3\kappa+2}{16(\kappa+1)}\left[r^2 - \frac{(3\kappa+2)}{2(\kappa+1)}r - \frac{\kappa}{2(\kappa+1)}\cdot\frac{1}{r}\right]\right\}Q_2(\mu) \quad [23]$$

where $a_i{}^j$ ($i = 1, 2, 3, 4$; $j = 0, 1$) are constants,

$$Q_n(\mu) = \int_{-1}^{1} P_n(x)\,dx$$

and P_n is the Legendre polynomial of degree n.

In seeking an expansion that holds for large r, it is necessary to find transformation of variables which will allow expansions well behaved as $r \to \infty$. The proper coordinate transformation for this problem is $\rho = rM$. If this transformation is introduced into [3] the result is

$$u = \frac{M^2}{\rho^2 \sin\theta} \cdot \frac{\partial \chi}{\partial \theta}, \qquad v = -\frac{M^2}{\rho \sin\theta} \cdot \frac{\partial \chi}{\partial \rho}.$$

Since u and v should not vanish as $M \to 0$, the variable $\Psi = M^2\chi$ is introduced to remove the dependence of $|M|$ from the velocities. The equation for the stream function Ψ in the outer region is

$$\frac{\partial}{\partial \tau}(\mathscr{A}^2\Psi) + \frac{Re}{|M|^2}\left[\frac{M}{\rho^2} \cdot \frac{\partial(\Psi, \mathscr{A}^2\Psi)}{\partial(\rho, \mu)} + \frac{2}{\rho^2} M\mathscr{A}^2\Psi \,.\, \mathscr{L}\Psi\right] = i\mathscr{A}^4\Psi, \qquad [24]$$

where

$$\mathscr{A}^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1-\mu^2}{\rho^2} \cdot \frac{\partial^2}{\partial \mu^2}; \qquad \mathscr{L} = \frac{\mu}{1-\mu^2} \cdot \frac{\partial}{\partial \rho} + \frac{1}{\rho} \cdot \frac{\partial}{\partial \mu}$$

The first-order solution Ψ_1 as $M \to 0$ is obtained to be

$$\Psi_1 = \frac{1}{2}\rho^2(1-\mu^2)\,e^{i\tau} + \frac{|M|(3\kappa+2)}{2(\kappa+1)}(1-\mu^2)\left[\left(\frac{1}{\rho}+1\right)e^{-\rho} - \frac{1}{\rho}\right]e^{i(\tau+\pi/4)}$$
$$+ \frac{|M|^2(3\kappa+2)^2}{6(\kappa+1)^2}(1-\mu^2)\left[\left(\frac{1}{\rho}+1\right)e^{-\rho} - \frac{1}{\rho}\right]e^{i(\tau+\pi/2)} + \cdots.$$

We suppose that

$$\Psi_2 = F_{20} + |M|F_{21} + |M|^2F_{22} + \cdots \qquad [25]$$

and

$$F_{22} = F_{22}^{(0)}(\rho, \mu) + F_{22}^{(2)}(\rho, \mu)\,e^{2i\tau}.$$

As in [20] one can show that $F_{20} = F_{21} = 0$. For $F_{22}^{(j)}$, $j = 0, 2$ we have

$$\mathscr{A}^4F_{22}^{(j)} - j\mathscr{A}^2F_{22}^{(j)} = \frac{(3k+2)}{2(\kappa+1)}Q_2(\mu)\,e^{-\rho}\left(1 + \frac{3}{\rho} + \frac{3}{\rho^2}\right). \qquad [26]$$

The resulting solutions for $F_{22}^{(0)}$ and $F_{22}^{(2)}$ are

$$F_{22}^{(0)} = \left[\mathscr{A}_{20} + \frac{B_{20}}{\rho^2} + \frac{(3\kappa+2)}{2(\kappa+1)}\left(1 + \frac{3}{\rho} + \frac{3}{\rho^2}\right)e^{-\rho}\right]Q_2(\mu),$$
$$F_{22}^{(2)} = \left[\frac{B_{22}}{\rho^2} + A_{22}\left(1 + \frac{3}{\sqrt{2}\rho} + \frac{3}{2\rho^2}\right)e^{-\sqrt{(2\rho)}} - \frac{3\kappa+2}{2(\kappa+1)}\left(1 + \frac{3}{\rho} + \frac{3}{\rho^2}\right)e^{-\rho}\right]Q_2(\mu). \qquad [27]$$

The matching procedure is based on the idea that χ and Ψ are different forms of the same function.

By means of the matching condition

$$\lim_{r\to\infty}[\chi_1 + \epsilon\chi_2 + \cdots] = \lim_{\rho\to 0}\left[\frac{1}{M^2}\psi_1 + \frac{\epsilon}{M^2}\psi_2 + \ldots\right],$$

we obtain

$$a_i^j = 0, \quad (j = 0, 2;\ I = 1, 2, 3, 4);$$

$$A_{20}=\frac{(3\kappa+2)}{4(\kappa+1)};\quad A_{22}=\frac{(3\kappa+2)}{2(\kappa+1)};\quad B_{20}=-\frac{3}{2}\cdot\frac{(3\kappa+2)}{(\kappa+1)};\quad B_{22}=\frac{3}{4}\cdot\frac{(3\kappa+2)}{(\kappa+1)}.$$

Therefore

$$\begin{aligned}\psi= & -\rho^2Q_1(\mu)\,e^{i\tau}-\frac{|M|Q_1(\mu)(3\kappa+2)}{(\kappa+1)}\left[\left(\frac{1}{\rho}+1\right)e^{-\rho}-\frac{1}{\rho}\right]e^{i(\tau+\pi/4)}\\ & -\frac{|M|^2(3\kappa+2)^2}{3(\kappa+1)^2}Q_1(\mu)\left[\left(\frac{1}{\rho}+1\right)e^{i(\tau+\pi/2)}+Re\frac{(3\kappa+2)}{2(\kappa+1)}\left[\frac{1}{2}-\frac{3}{\rho^2}+\left(1+\frac{3}{\rho}+\frac{3}{\rho^2}\right)e^{-\rho}\right]Q_2(\mu)\right.\\ & +Re\frac{(3\kappa+2)}{2(\kappa+1)}\left[\frac{3}{2\rho^2}+\left(1+\frac{3}{\rho\sqrt{2}}+\frac{3}{2\rho^2}\right)e^{-\sqrt{(2\rho)}}-\left(1+\frac{3}{\rho}+\frac{3}{\rho^2}\right)e^{-\rho}\right]Q_2(\mu)\,e^{2i\tau}+\cdots\\ \chi= & -\frac{Q_1(\mu)}{2}\left[2r^2-\frac{(3\kappa+2)}{(\kappa+1)}r+\frac{\kappa}{(\kappa+1)}\cdot\frac{1}{r}\right]e^{i\tau}\\ & -\frac{|M|Q_1(\mu)}{3}\cdot\frac{(3\kappa+2)}{(\kappa+1)^2}\left[-\frac{3\kappa+2}{2}\cdot r+\frac{\kappa}{2\tau}+(\kappa+1)r^2\right]e^{i(\tau+\pi/4)}\\ & +\frac{|M|^2Q_1(\mu)}{504(\kappa+1)^3}\left\{63(3\kappa+2)(\kappa+1)^2r^3-56(3\kappa+2)^2(\kappa+1)r^2\right.\\ & \left.+[12\gamma(\kappa+1)+14(3\kappa+2)(9\kappa^2+12\kappa+5)]r-[12\gamma(\kappa+1)+7(3\kappa+2)(3\kappa^2+2\kappa+3)]\frac{1}{r}\right\}e^{i(\tau+\pi/2)}\\ & +\frac{Re}{16}\cdot\frac{(3\kappa+2)}{(\kappa+1)}\left\{\frac{\kappa(5\kappa+4)}{10(\kappa+1)^2}+\frac{\kappa(5\kappa+6)}{10(\kappa+1)^2}\cdot\frac{1}{r^2}+r^2-\frac{3\kappa+2}{2(\kappa+1)}r-\frac{\kappa}{2(\kappa+1)}\cdot\frac{1}{r}\right\}Q_2(\mu)(1+e^{2i\tau})+\cdots\end{aligned}\qquad [28]$$

Similarly, for the region inside the drop, we seek $\hat{\chi}_2$ in the form

$$\hat{\chi}_2=\hat{\phi}_{20}+|M|\hat{\phi}_{21}+|M|^2\hat{\phi}_{22}+\cdots$$

where

$$\hat{\phi}_{22}=\hat{\phi}_{22}^{(0)}+\hat{\phi}_{22}^{(2)}e^{2i\tau}.$$

Straight forward but tedious computations, which will not be reported here, yield

$$\hat{\phi}_{20}=\hat{\phi}_{21}=0;\quad \hat{\phi}_{22}^{(0)}=\hat{\phi}_{22}^{(2)}=Q_2(\mu)\frac{(4\kappa+5)(3\kappa+2)}{160(\kappa+1)^3}(r^5-r^3).$$

Hence for the $\hat{\chi}$ we have

$$\begin{aligned}\hat{\chi}= & \frac{Q_1(\mu)}{2(\kappa+1)}\cdot(r^2-r^4)\,e^{i\tau}+\frac{|M|Q_1(\mu)}{6}\cdot\frac{(3\kappa+2)}{(\kappa+1)^2}\cdot(r^2-r^4)\,e^{i(\tau+\pi/4)}-\frac{|M|^2Q_1(\mu)}{504(\kappa+1)^3\kappa}\\ & \times[9\gamma(\kappa+1)^2r^6-2(3\kappa+2)(3\gamma\kappa+3\gamma+17\kappa)r^7+(9\gamma+30\gamma\kappa+21\gamma^2\kappa+78\kappa^2+56\kappa)r^2]\,e^{i(\tau+\pi/2)}\\ & +ReQ_2(\mu)\frac{(3\kappa+2)(4\kappa+5)}{160(\kappa+1)^3}(r^5-r^3)(1+e^{2i\tau})+\cdots.\end{aligned}\qquad [29]$$

As a matter of fact, it is interesting to note that for the limiting case of small gas bubbles ($\kappa\to 0$, $\gamma\to 0$) we get Davidson & Riley's (1971) solution for the region $r\geq 1$, whereas for very viscous drops ($\kappa\to\infty$)

$$\begin{aligned}\Psi= & -\rho^2Q_1(\mu)\,e^{i\tau}-3|M|Q_1(\mu)\left[\left(\frac{1}{\rho}+1\right)e^{-\rho}-\frac{1}{\rho}\right]e^{i(\tau+\pi/4)}-3|M|^2Q_1(\mu)\left[\left(\frac{1}{\rho}+1\right)e^{-\rho}-\frac{1}{\rho}\right]e^{i(\tau+\pi/2)}\\ & +Re\cdot\frac{3}{2}\left[\frac{1}{2}-\frac{3}{\rho^2}+\left(1+\frac{3}{\rho}+\frac{3}{\rho^2}\right)e^{-\rho}\right]Q_2(\mu)+Re\cdot\frac{3}{2}\left[\frac{3}{2\rho^2}+\left(1+\frac{3}{\rho\sqrt{2}}+\frac{3}{2\rho^2}\right)e^{-\sqrt{(2)}\rho}\right.\\ & \left.-\left(1+\frac{3}{\rho}+\frac{3}{\rho^2}\right)e^{-\rho}\right]Q_2(\mu)\,e^{2i\tau}+\cdots\end{aligned}\qquad [30]$$

$$\chi = -\frac{Q_1(\mu)}{2}\left[2r^2 - 3r + \frac{1}{r}\right] e^{i\tau} - |M| Q_1(\mu)\left[-\frac{3}{2}r + \frac{1}{2\tau} + r^2\right] e^{i(\tau+\pi/4)}$$
$$+\frac{|M|^2}{8} Q_1^2(\mu)\left\{3r^3 - 8r^2 + 6r - \frac{1}{\tau}\right\} e^{i(\tau+\pi/2)} + Re\cdot\frac{3}{16}\left\{\frac{1}{2} + \frac{1}{2r^2} + r^2 - \frac{3}{2}r\right.$$
$$\left.-\frac{1}{2\tau}\right\} Q_2(\mu)(1 + e^{2i\tau}) + \cdots. \qquad [32]$$
$$\hat{\chi} = 0.$$

The significance of the viscosity of the drop material to the steady state streaming which accompanies the oscillation one can see from the formulae:

$$\Psi_2^{(s)} = \frac{(3\kappa+2)}{2(\kappa+1)}\left[\frac{1}{2} - \frac{3}{\rho^2} + \left(1 + \frac{3}{\rho} + \frac{3}{\rho^2}\right) e^{-\rho}\right] Q_2(\mu), \qquad [33]$$

$$\chi_2^{(s)} = \frac{(3\kappa+2)}{16(\kappa+1)}\left[\frac{\kappa(5\kappa+4)}{10(\kappa+1)^2} + \frac{\kappa(5\kappa+6)}{10(\kappa+1)^2}\cdot\frac{1}{r^2} + r^2 - \frac{(3\kappa+2)}{2(\kappa+1)}r - \frac{\kappa}{2(\kappa+1)}\cdot\frac{1}{r}\right] Q_2(\mu), \qquad [34]$$

$$\hat{\chi}_2^{(5)} = \frac{(4\kappa+5)(3\kappa+2)}{160(\kappa+1)^3}(r^5 - r^3) Q_2(\mu). \qquad [35]$$

It is of interest to calculate the drag on the drop. We have

$$F_D = \frac{drag}{a^2\rho V_\infty^2} = \frac{2n}{Re}\int_0^\pi [p_{rr}|_{r=1}\cos\theta - p_{r\theta}|_{r=1}\sin\theta]\sin\theta\, d\theta \qquad [36]$$

where

$$p_{r\theta} = -p + 2\frac{\partial u}{\partial r},$$

$$p_{r\theta} = \frac{1}{r}\cdot\frac{\partial u}{\partial r} + \frac{\partial v}{\partial r} - \frac{V}{r}. \qquad [37]$$

By manipulating the radial and tangential momentum equations, we can derive an expression for the pressure

$$p = p_0 - \frac{(3\kappa+2)}{2r^2(\kappa+1)}\left[1 + \frac{(3\kappa+2)}{3(\kappa+1)}M\right]\cos\theta\, e^{i\tau}. \qquad [38]$$

Using [36]–[38] we get

$$F_D = \frac{2\pi}{Re}\cdot\frac{(3\kappa+2)}{(\kappa+1)}\left[1 + \frac{(3\kappa+2)}{3(\kappa+1)}M\right] e^{i\tau} \qquad [39]$$

This expression is (in some sense) a generalization of the Taylor & Acrivos' (1964) formula [10a] for the case of an oscillating fluid drop immersed in another fluid.

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